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HYPERBOLIC LENGTHS AND CONFORMAL EMBEDDINGS OF RIEMANN SURFACES

BY

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ABSTRACT

Let h be a homeomorphic bijection between hyperbolic Riemann surfaces R and R'. If there is a conformal mapping of R into R' homotopic to h, then for any hyperbolic geodesic c on R the length of the hyperbolic geodesic freely homotopic to the image $h(c)$ is less than or equal to the hyperbolic length of c. We show that the converse is not necessarily true.

1. Introduction

Let R and R' be noncompact Riemann surfaces. Suppose that they are homeomorphic to each other, and fix a homeomorphism h of R onto R'. Then we are interested in the condition that

[C] *there exists a conformal mapping of R into R' which is homotopic to h.*

Here, by a conformal mapping we mean a holomorphic injection; it is not necessarily surjective.

We look for conditions which imply or are implied by condition [C]. First we give an elementary condition necessary for [C] in terms of hyperbolic lengths, and then ask whether it is also sufficient for [C] or not.

In general, let $\mathcal{S}[W]$ be the set of free homotopy classes of closed curves on a Riemann surface W. If W is hyperbolic (that is, the universal covering surface of W is conformally equivalent to the unit disk $\mathbb{D}$), then for $c \in \mathcal{S}[W]$ we denote

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by $l_W(c)$ the infimum of the hyperbolic lengths of curves in the homotopy class c, where the hyperbolic metric is normalized to have curvature -1. We simply call $l_W(c)$ the hyperbolic length of c.

Now, the homeomorphism $h\colon R \to R'$ induces a bijection h_* of $\mathcal{S}[R]$ onto $\mathcal{S}[R']$. If R and R' are hyperbolic and condition [C] is valid, then h_* decreases the hyperbolic lengths:

[H] $l_{R'}(h_*(c)) \leq l_R(c)$ *for all* $c \in \mathcal{S}[R]$.

Thus condition [C] implies condition [H]. If R and R' are planar and doubly connected, then the converse is also true. In this paper we show that in general this is not always the case.

To state our main result we need one more definition. A boundary component p of W is called **border-like** if there is a doubly connected planar subdomain D of W of finite modulus such that $D \cup \{p\}$ is a neighborhood of p in the Kerékjártó–Stoïlow compactification of W. (See Ahlfors–Sario [2, I.36] for the definitions of a boundary component and the Kerékjártó–Stoïlow compactification.) If W is a subdomain of a compact Riemann surface $\tilde{W}$, then W has a border-like boundary component if and only if $\tilde{W} \setminus W$ has a connected component which contains more than one point and which is isolated from the other components.

THEOREM 1.1: *Let R be a noncompact Riemann surface of positive finite genus. Assume that R has a border-like boundary component. Then there is a Riemann surface R' together with a homeomorphism h of R onto R' such that* [H] *is valid while* [C] *is not.*

This is a generalization of our previous result [11, Theorem 5.2], where we treated the case of a torus with a hole.

For the sake of comparison we discuss another elementary condition similar to [H]. For $c \in \mathcal{S}[W]$ let $\lambda_W(c)$ denote the extremal length of the curve family c. If condition [C] holds, then h_* also decreases the extremal lengths:

[E] $\lambda_{R'}(h_*(c)) \leq \lambda_R(c)$ *for all* $c \in \mathcal{S}[R]$.

Hence [E] is another necessary condition for [C]. Of course, for doubly connected planar Riemann surfaces other than the punctured plane, [E] is also sufficient for [C]. We know a less trivial case where condition [E] is sufficient for [C]:

THEOREM 1.2 ([11, Theorem 5.1]): *If R and R' are homeomorphic to a once-punctured torus, then for any homeomorphism h of R onto R' condition* [E] *implies condition* [C].

Thus in the space of Riemann surfaces homeomorphic to a once-punctured torus, condition [E] is necessary and sufficient for condition [C] while [H] is only a necessary, not sufficient, condition for [C]. It is not known whether Theorem 1.2 is still valid for more general classes of Riemann surfaces.

In §2 we review the results of Shiba [18], who investigated the space $M(R,\chi)$ of compact continuations of a marked noncompact Riemann surface (R,χ) of positive finite genus g. His results give us necessary conditions for [C] in terms of period matrices. In §3 we introduce the space $\mathfrak{M}(R,\chi)$ of marked compact Riemann surfaces of genus g into which there is a holomorphic mapping of (R,χ) homotopic to a homeomorphic injection. We then show that $M(R,\chi)$ is a proper subset of $\mathfrak{M}(R,\chi)$ if R has a border-like boundary component. The proof of Theorem 1.1 finishes in §5. A Riemann surface R' possessing the properties in the theorem is obtained as a subdomain of a compact Riemann surface in $\mathfrak{M}(R,\chi) \setminus M(R,\chi)$.

2. Compact continuations of a noncompact Riemann surface

We begin with the definition of a marking of a Riemann surface R of positive finite genus g. First suppose that R is compact. As is well known, a canonical dissection of R gives rise to an ordered set $\chi = \{a_1, b_1, \ldots, a_g, b_g\}$ of elements of the fundamental group $\pi_1(R,p)$ with base point at $p \in R$ such that $a_1, b_1, \ldots, a_g, b_g$ generate $\pi_1(R,p)$ with the single defining relation

$$\prod_{j=1}^{g}(a_j b_j a_j^{-1} b_j^{-1}) = 1.$$

Such an ordered set χ is called a **marking** of R with base point at p.

Next, if R is noncompact, then it is homeomorphically embedded into a compact Riemann surface $\tilde{R}$ of the same genus. A homeomorphism $\tilde{h}$ of R into $\tilde{R}$ induces a homomorphism $\tilde{h}_*\colon \pi_1(R,p) \to \pi_1(\tilde{R}, \tilde{h}(p))$ for any $p \in R$. We then say that an ordered set $\chi = \{a_1, b_1, \ldots, a_g, b_g\}$ of elements of $\pi_1(R,p)$ is a **marking** of R if

$$\tilde{h}_*(\chi) = \{\tilde{h}_*(a_1), \tilde{h}_*(b_1), \ldots, \tilde{h}_*(a_g), \tilde{h}_*(b_g)\}$$

is a marking of $\tilde{R}$. This definition does not depend on a particular choice of $\tilde{R}$ or $\tilde{h}$. Note that χ is *not* a set of generators of the fundamental group of R unless R has exactly one boundary component.

Now, let R be an arbitrary (compact or noncompact) Riemann surface of genus g. Every curve c joining p to p' on R induces an isomorphism ω_c of $\pi_1(R,p)$ onto

$\pi_1(R,p')$ in a well-known manner. Let $\chi = \{a_j, b_j\}_{j=1}^g$ and $\chi' = \{a'_j, b'_j\}_{j=1}^g$ be two markings of R with base points at p and p', respectively. If there exists a curve c from p to p' such that $\omega_c(a_j) = a'_j$ and $\omega_c(b_j) = b'_j$ for $j = 1, \ldots, g$, then we say that the markings χ and χ' are equivalent to each other.

A **marked Riemann surface** of genus g is a pair (R, χ), where R is a Riemann surface of genus g and χ is a marking of R. Let (R', χ') be another marked Riemann surface of the same genus. Every continuous mapping f of R into R' induces a homomorphism f_*: $\pi_1(R,p) \to \pi_1(R', f(p))$ for any $p \in R$. If $f_*(\chi)$ is a marking of R' equivalent to χ', then we say that f is a continuous mapping of (R, χ) into (R', χ') and use the notation f: $(R, \chi) \to (R', \chi')$. If, in addition, f is holomorphic (resp. conformal, injective, etc.), then we say that f: $(R, \chi) \to (R', \chi')$ is holomorphic (resp. conformal, injective, etc.). If there exists a conformal bijection of (R, χ) onto (R', χ'), then these marked Riemann surfaces are said to be conformally equivalent to each other. The conformal equivalence class of (R, χ) is denoted by $[R, \chi]$. Also, if R is a subdomain of R' and the marking of R' induced by χ in the natural way is equivalent to χ', then we write $(R, \chi) \subset (R', \chi')$. In this case the inclusion mapping $R \hookrightarrow R'$ is a conformal mapping of (R, χ) into (R', χ').

The **Teichmüller space** T_g of genus g is the set of conformal equivalence classes of marked *compact* Riemann surfaces of genus g. It is a complex manifold of dimension $3g - 3$ if $g > 1$ and of dimension 1 if $g = 1$. For $[\tilde{R}, \tilde{\chi}] \in T_g$ with $\tilde{\chi} = \{\tilde{a}_j, \tilde{b}_j\}_{j=1}^g$ we set

$$\pi_{jk}[\tilde{R}, \tilde{\chi}] = \int_{\tilde{b}_j} \tilde{\varphi}_k, \quad j, k = 1, \ldots, g,$$

where $\tilde{\varphi}_k$'s are the unique holomorphic differentials on $\tilde{R}$ such that $\int_{\tilde{a}_j} \tilde{\varphi}_k = \delta_{jk}$ for $j, k = 1, \ldots, g$. Here δ_{jk} stands for the Kronecker delta. The $g \times g$ matrix $\Pi[\tilde{R}, \tilde{\chi}] := (\pi_{jk}[\tilde{R}, \tilde{\chi}])$ is called the **normalized period matrix** of $[\tilde{R}, \tilde{\chi}]$. It belongs to the Siegel upper half space $\mathfrak{S}_g$, that is, $\Pi[\tilde{R}, \tilde{\chi}]$ is a symmetric $g \times g$ matrix with positive definite imaginary part. Moreover, Π: $[\tilde{R}, \tilde{\chi}] \mapsto \Pi[\tilde{R}, \tilde{\chi}]$ is a holomorphic mapping of T_g into $\mathfrak{S}_g$. In particular, the diagonal entries π_{kk} are holomorphic mappings of T_g into the upper half plane $\mathbb{H}$.

M. Shiba has been studying the space of conformal embeddings of a noncompact Riemann surface of finite genus into compact Riemann surfaces of the same genus. In the rest of this section we summarize some of his results. For the proofs we refer to [18]. Though he considered pairs of Riemann surfaces and canonical homology bases, it is easy to verify our version of his results.

For a marked noncompact Riemann surface (R, χ) of genus g we denote by

$M(R,\chi)$ the set of points $[\tilde{R},\tilde{\chi}]$ of T_g for which (R,χ) can be conformally embedded into $(\tilde{R},\tilde{\chi})$. Shiba investigated the behavior of Π in $M(R,\chi)$.

PROPOSITION 2.1: *For each $k = 1,\ldots,g$ the image $\pi_{kk}(M(R,\chi))$ of $M(R,\chi)$ under the mapping π_{kk} is a closed disk or a point in $\mathbb{H}$.*

Shiba first showed in [18] that there is a closed disk $\Delta_k \subset \mathbb{H}$ such that $\partial\Delta_k \subset \pi_{kk}(M(R,\chi)) \subset \Delta_k$, and then, in a joint work with Schmieder (see [16, Theorem 3]), he showed that $\pi_{kk}(M(R,\chi)) = \Delta_k$ for R of topologically finite type. The general result has been established in [12, Theorem 10.1]. For the proof of Theorem 1.1, however, the weaker version [18] is sufficient.

PROPOSITION 2.2: *If $R \in O_{AD}$, then $\pi_{kk}(M(R,\chi))$ degenerates to a point for all k. Conversely, if $\pi_{kk}(M(R,\chi))$ is a singleton for some k, then $R \in O_{AD}$.*

Consequently, if R has a border-like boundary component, then R does not belong to O_{AD} and so $\pi_{kk}(M(R,\chi))$ is a closed disk of positive radius for any k.

Shiba introduced the notion of canonical hydrodynamic continuation. A canonical hydrodynamic continuation of (R,χ) is a conformal embedding $\tilde{\iota}$ of (R,χ) into a marked compact Riemann surface $(\tilde{R},\tilde{\chi})$ of genus g such that a holomorphic differential of special character has a holomorphic extension to $\tilde{R}$, where R is identified with $\tilde{\iota}(R)$. He proved that if $\tilde{\iota}$: $(R,\chi) \to (\tilde{R},\tilde{\chi})$ is a canonical hydrodynamic continuation, then $\pi_{kk}[\tilde{R},\tilde{\chi}]$ lies on the boundary of $\pi_{kk}(M(R,\chi))$ for some k, and vice versa. Here we discuss only canonical hydrodynamic continuations corresponding to the bottom point of $\pi_{kk}(M(R,\chi))$. For the precise definition and more general properties of canonical hydrodynamic continuations, see [18] and [19].

For each $k = 1,\ldots,g$ there uniquely exists a holomorphic differential φ_k on R whose imaginary part is a distinguished harmonic differential in the sense of Ahlfors (for the definition, see Ahlfors–Sario [2, V.21D]) and which satisfies $\int_{a_j}\varphi_k = \delta_{jk}$ for $j = 1,\ldots,g$. We call φ_k the **k-th canonical holomorphic differential** of (R,χ). A conformal mapping $\tilde{\iota}$ of (R,χ) into a marked compact Riemann surface $(\tilde{R},\tilde{\chi})$ of the same genus is called a **canonical hydrodynamic continuation** of (R,χ) with respect to φ_k if

(2.1) the area of $\tilde{R}\setminus\tilde{\iota}(R)$ vanishes,

(2.2) the pull-back of φ_k via $\tilde{\iota}^{-1}$: $\tilde{\iota}(R) \to R$ extends to a holomorphic differential $\tilde{\varphi}_k$ on $\tilde{R}$, and

(2.3) every connected component of $\tilde{R}\setminus\tilde{\iota}(R)$ containing more than one point consists of finitely many analytic arcs along which $\operatorname{Im}\tilde{\varphi}_k = 0$ (and possibly zeros of $\tilde{\varphi}_k$).

Canonical hydrodynamic continuations are characterized by the following extremal property:

PROPOSITION 2.3: *Let $[\tilde{R},\tilde{\chi}] \in M(R,\chi)$ and let $\tilde{\iota}$: $(R,\chi) \to (\tilde{R},\tilde{\chi})$ be a conformal embedding. Then $\tilde{\iota}$ is a canonical hydrodynamic continuation with respect to the k-th canonical holomorphic differential of (R,χ) if and only if $\pi_{kk}[\tilde{R},\tilde{\chi}]$ has the smallest imaginary part in $\pi_{kk}(M(R,\chi))$:*

$$\operatorname{Im}\pi_{kk}[\tilde{R},\tilde{\chi}] = \min \operatorname{Im}\pi_{kk}(M(R,\chi)). \tag{2.4}$$

The following proposition is an immediate consequence of Proposition 2.3 and condition (2.1).

PROPOSITION 2.4: *Let $[\tilde{R},\tilde{\chi}]$ be a point of $M(R,\chi)$. If (2.4) holds, then $\tilde{R}\setminus\tilde{\iota}(R)$ has a vanishing area for any conformal mapping $\tilde{\iota}$ of (R,χ) into $(\tilde{R},\tilde{\chi})$.*

There may be infinitely many points $[\tilde{R},\tilde{\chi}] \in M(R,\chi)$ which satisfy (2.4). In other words we cannot assert the uniqueness of a canonical hydrodynamic continuation with respect to the k-th canonical holomorphic differential φ_k on (R,χ). In view of condition (2.2) we see that φ_k has at most $2g-2$ zeros in R. If it has $2g-2$ zeros in R, then the uniqueness assertion holds in the sense that there is exactly one $[\tilde{R},\tilde{\chi}] \in M(R,\chi)$ satisfying (2.4) and that (R,χ) is conformally embedded into $(\tilde{R},\tilde{\chi})$ uniquely up to conformal automorphisms of $(\tilde{R},\tilde{\chi})$.

3. Holomorphic mappings and conformal mappings

In the present section we consider holomorphic mappings of a marked noncompact Riemann surface (R,χ) of positive finite genus g into marked compact Riemann surfaces of the same genus. Let $\mathfrak{M}(R,\chi)$ denote the set of $[\tilde{R},\tilde{\chi}] \in T_g$ such that there is a holomorphic mapping of (R,χ) into $(\tilde{R},\tilde{\chi})$ which is homotopic to a homeomorphism of (R,χ) into $(\tilde{R},\tilde{\chi})$. Obviously, we have

$$M(R,\chi) \subset \mathfrak{M}(R,\chi).$$

We investigate whether $M(R,\chi)$ coincides with $\mathfrak{M}(R,\chi)$ or not. First of all we remark the following

PROPOSITION 3.1: *If R is obtained from a compact Riemann surface by deleting a discrete set, then $M(R,\chi) = \mathfrak{M}(R,\chi)$ whenever $g > 1$.*

Proof: Let $\bar{R}$ be the compact Riemann surface from which R is obtained. If $[\tilde{S},\tilde{\sigma}]$ is a point of $\mathfrak{M}(R,\chi)$ and $\tilde{f}$ is a holomorphic mapping of (R,χ) into $(\tilde{S},\tilde{\sigma})$, then by Huber [5, Satz 2 in §6] (see also Marden–Richards–Rodin [9, Theorem 1]) we know that $\tilde{f}$ is extended to a holomorphic mapping of $\bar{R}$ into $\tilde{S}$. Since these Riemann surfaces are compact and have the same genus, the extended holomorphic mapping is indeed a conformal bijection. This implies that $\mathfrak{M}(R, \chi)$ is a singleton, and the proof is complete. ∎

The assumption that $g > 1$ is essential as the following proposition shows:

PROPOSITION 3.2: *If (R, χ) is a marked noncompact Riemann surface of genus one, then $\mathfrak{M}(R, \chi)$ is identical with the Teichmüller space T_1 of genus one.*

Proof: We proceed as in [11, Proposition 2.2] (see also [17] and the references quoted there). Every element of T_1 is of the form $[\mathbb{C}/G_\tau, \chi_\tau]$ with $\tau \in \mathbb{H}$, where G_τ is the lattice generated by 1 and τ and where χ_τ is composed of the images of the oriented segments joining 0 to 1 and to τ under the natural projection $\pi_\tau \colon \mathbb{C} \to \mathbb{C}/G_\tau$.

Now, we may assume that $(R, \chi) \subset (\mathbb{C}/G_\sigma, \chi_\sigma)$ for some $\sigma \in \mathbb{H}$. For each $\tau \in \mathbb{H}$, by a theorem of Behnke–Stein, there is a holomorphic differential φ on R satisfying that $\int_a \varphi = 1$, $\int_b \varphi = \tau$, where $\chi = \{a, b\}$. Furthermore, we require that $\int_c \varphi = 0$ for any dividing cycles c on R. Then its abelian integral $\Phi(p) = \int^p \varphi$ induces a (single-valued) holomorphic mapping $f := \pi_\tau \circ \Phi$ of (R, χ) into $(\mathbb{C}/G_\tau, \chi_\tau)$. It is lifted to a holomorphic mapping F of $\pi_\tau^{-1}(R)$ into $\mathbb{C}$ such that $F(z+1) = F(z)+1$ and $F(z+\sigma) = F(z)+\tau$. Let A be the affine mapping of $\mathbb{C}$ onto itself carrying 0, 1, σ to 0, 1, τ, respectively. It induces a homeomorphism α of R into $\mathbb{C}/G_\tau$. Since the continuous mapping $H \colon \pi_\tau^{-1}(R) \times [0,1] \to \mathbb{C}$ defined by $H(z,t) = tF(z) + (1-t)A(z)$ gives a homotopy connecting f and α, we know that $[\mathbb{C}/G_\tau, \chi_\tau]$ belongs to $\mathfrak{M}(R, \chi)$. ∎

Proposition 3.2 shows in particular that $M(R, \chi)$ is always a proper subset of $\mathfrak{M}(R, \chi)$ if $g = 1$. In the case where $g > 1$, the same thing holds under an additional assumption on R (cf. Proposition 3.1).

THEOREM 3.1: *Let (R, χ) be a marked noncompact Riemann surface of positive finite genus. If R has a border-like boundary component, then $M(R, \chi) \subsetneqq \mathfrak{M}(R, \chi)$.*

For the proof we prepare a lemma, which will also play an important role in the proof of Theorem 1.1.

LEMMA 3.1: *Let (R, χ) be a marked noncompact Riemann surface of positive finite genus with a border-like boundary component. Then there is a marked* *noncompact Riemann surface (R', χ') of the same genus together with a homeomorphic bijection h: $(R, \chi) \to (R', \chi')$ such that*

(i) $M(R', \chi') \setminus M(R, \chi) \neq \varnothing$, *and*

(ii) *there is a holomorphic mapping of (R, χ) into (R', χ') homotopic to h.*

The proof of Lemma 3.1 is fairly long and we will give it in the next section. Here, assuming the validity of Lemma 3.1, we prove Theorem 3.1.

Proof of Theorem 3.1: Let (R', χ') be a marked noncompact Riemann surface of the same genus possessing properties (i) and (ii) of Lemma 3.1. We deduce from property (ii) that $\mathfrak{M}(R', \chi') \subset \mathfrak{M}(R, \chi)$. Since $M(R', \chi') \subset \mathfrak{M}(R', \chi')$, we obtain by property (i) that $\mathfrak{M}(R, \chi) \setminus M(R, \chi) \neq \varnothing$, as desired. ∎

4. Proof of Lemma 3.1

In this section we give a proof of Lemma 3.1. Let $\tilde{\iota}$: $(R, \chi) \to (\tilde{R}, \tilde{\chi})$ be a canonical hydrodynamic continuation with respect to the first canonical holomorphic differential φ of (R, χ). Thus $[\tilde{R}, \tilde{\chi}]$ is a point of $M(R, \chi)$ such that $\pi_{11}[\tilde{R}, \tilde{\chi}]$ has the smallest imaginary part in $\pi_{11}(M(R, \chi))$:

$$\operatorname{Im} \pi_{11}[\tilde{R}, \tilde{\chi}] = \min \operatorname{Im} \pi_{11}(M(R, \chi)) \tag{4.1}$$

(see Proposition 2.3). Identifying R with $\tilde{\iota}(R)$, we may assume from the outset that R is a subdomain of $\tilde{R}$. Since R has a border-like boundary component, there is a connected component C of $\tilde{R} \setminus R$ which is isolated from the other components and which contains more than one point. It follows from (2.2) and (2.3) that φ is extended to a holomorphic differential $\tilde{\varphi}$ on $\tilde{R}$ whose imaginary part vanishes along C.

Let p_0 be a point of C that is not a zero of $\tilde{\varphi}$. We can find a parametric disk U in $\tilde{R}$ around p_0 with conformal bijection z: $U \to \mathbb{D}$, $z(p_0) = 0$, such that

(a) z is analytic and homeomorphic on the closure $\bar{U}$ of U,

(b) $\bar{U} \subset R \cup C$, that is, $\bar{U}$ never meets the other components of $\tilde{R} \setminus R$,

(c) $\bar{U}$ contains no zeros of $\tilde{\varphi}$, and

(d) $C_0 := \{p \in U \colon |\operatorname{Re} z(p)| \leq 1/2 \text{ and } \operatorname{Im} z(p) = 0\} \subset C$.

We define $S_0 = \tilde{R} \setminus C_0$. More generally, we set $S_\varepsilon = \tilde{R} \setminus C_\varepsilon$ for $\varepsilon \in (0, \sqrt{2} - 1)$, where

$$C_\varepsilon = C_0 \cup \{p \in U \colon |z(p) + i| \leq 1 + \varepsilon \text{ and } \operatorname{Im} z(p) \geq 0\}.$$

Clearly, S_ε, where $0 \leq \varepsilon < \sqrt{2} - 1$, are noncompact Riemann surfaces of genus g with exactly one boundary component. We choose a marking σ_ε of S_ε so that

$$(R, \chi) \subset (S_0, \sigma_0) \subset (\tilde{R}, \tilde{\chi}) \quad \text{and} \quad (S_\varepsilon, \sigma_\varepsilon) \subset (S_0, \sigma_0).$$

The latter inclusion relation yields

$$M(S_0,\sigma_0) \subset M(S_\varepsilon,\sigma_\varepsilon). \tag{4.2}$$

Let $\tilde{\eta}_\varepsilon$: $(S_\varepsilon,\sigma_\varepsilon) \to (\tilde{S}_\varepsilon,\tilde{\sigma}_\varepsilon)$ be a canonical hydrodynamic continuation of $(S_\varepsilon,\sigma_\varepsilon)$ with respect to the first canonical holomorphic differential φ_ε of $(S_\varepsilon,\sigma_\varepsilon)$. By Proposition 2.3 we have

$$\operatorname{Im}\pi_{11}[\tilde{S}_\varepsilon,\tilde{\sigma}_\varepsilon] = \min \operatorname{Im}\pi_{11}(M(S_\varepsilon,\sigma_\varepsilon)). \tag{4.3}$$

We first show that $\tilde{\eta}_\varepsilon$ is extended to a holomorphic mapping of S_0 into $\tilde{S}_\varepsilon$ whenever ε is sufficiently small. To this end we examine a branch Φ_ε of the abelian integral $\int^p \varphi_\varepsilon$ of φ_ε in the doubly connected domain $D_\varepsilon := S_\varepsilon \cap U = U \setminus C_\varepsilon$. The boundary ∂D_ε of D_ε in $\tilde{R}$ is composed of the "outer" boundary ∂U and the "inner" boundary ∂S_ε. Note that $D_\varepsilon \subset D_0$ and $S_\varepsilon \cup D_0 = S_0$. Since φ_ε is the pull-back of a holomorphic differential $\tilde{\psi}_\varepsilon$ on $\tilde{S}_\varepsilon$ by $\tilde{\eta}_\varepsilon$ (see (2.2)), the integral Φ_ε is the pull-back, via $\tilde{\eta}_\varepsilon$, of a branch $\tilde{\Psi}_\varepsilon$ of the abelian integral of $\tilde{\psi}_\varepsilon$ in the simply connected domain $\tilde{\Delta}_\varepsilon := \tilde{\eta}_\varepsilon(D_\varepsilon) \cup (\tilde{S}_\varepsilon \setminus \tilde{\eta}_\varepsilon(S_\varepsilon))$. In particular, Φ_ε is single-valued in D_ε. Observe that φ_0 coincides with the restriction of $\tilde{\varphi}$ to S_0 since $\operatorname{Im}\tilde{\varphi}$ vanishes along C_0. It follows from (c) that φ_0 has $2g-2$ zeros in $S_0 \setminus D_0$. Hence we have

$$[\tilde{S}_0,\tilde{\sigma}_0] = [\tilde{R},\tilde{\chi}]. \tag{4.4}$$

Since φ_ε converges to φ_0 as $\varepsilon \downarrow 0$ uniformly on each compact subset of S_0, for sufficiently small ε there are $2g-2$ zeros of φ_ε in $S_\varepsilon \setminus D_\varepsilon$ (and no zeros in D_ε). This implies that $\tilde{\psi}_\varepsilon$ has no zeros in $\tilde{\Delta}_\varepsilon$. Consequently, $\tilde{\Psi}_\varepsilon$ is locally univalent in $\tilde{\Delta}_\varepsilon$. Since Φ_ε is the pull-back of $\tilde{\Psi}_\varepsilon$, we know that the behavior of Φ_ε near the boundary ∂S_ε reflects the boundary behavior of $\tilde{\eta}_\varepsilon$. Since the imaginary part of Φ_ε is constant on ∂S_ε, the reflection principle enables us to extend Φ_ε across $\partial S_\varepsilon \cap D_0$ to a holomorphic function on D_0. This implies that $\tilde{\eta}_\varepsilon$ is extended to a holomorphic mapping of S_0 into $\tilde{S}_\varepsilon$, which will be denoted by the same symbol $\tilde{\eta}_\varepsilon$.

Next we claim that $\tilde{\eta}_\varepsilon(S_0)$ is a *proper* subdomain of $\tilde{S}_\varepsilon$ if ε is sufficiently small. Since $|\tilde{\Psi}_\varepsilon|$ attains its maximum on $\partial\tilde{\Delta}_\varepsilon$, its pull-back $|\Phi_\varepsilon|$ is bounded by the maximum on the "outer" boundary ∂U of D_ε. If Φ_ε is normalized to vanish at a fixed point q of D_0, then Φ_ε converges to Φ_0 as $\varepsilon \downarrow 0$ uniformly on each compact subset of $D_0 \cup \partial U$. We infer that $\{\Phi_\varepsilon\}_\varepsilon$ is uniformly bounded. Therefore, again with the aid of the reflection principle, we see that $\Phi_\varepsilon \to \Phi_0$ as $\varepsilon \downarrow 0$ uniformly on each compact subset of

$$\{p \in U: \operatorname{Im} z(p) \geq 0\} \setminus \{p_0\}.$$

Thus, for sufficiently small $\varepsilon > 0$, the image of $\partial S_\varepsilon \cap D_0$ under $\tilde{\eta}_\varepsilon$ is so small that it cannot cover all $\tilde{S}_\varepsilon \setminus \tilde{\eta}_\varepsilon(S_\varepsilon)$. Hence $\tilde{\eta}_\varepsilon$: $S_0 \to \tilde{S}_\varepsilon$ is *not* surjective, as asserted.

Finally, we fix $\varepsilon > 0$ such that $\tilde{\eta}_\varepsilon$: $S_0 \to \tilde{S}_\varepsilon$ is not surjective. Let $\tilde{E}$ be a connected component of $\tilde{S}_\varepsilon \setminus \tilde{\eta}_\varepsilon(S_0)$, and set

$$R' = \left\{\tilde{\eta}_\varepsilon\Big(R \cup (C \setminus C_0)\Big) \cup \tilde{\Delta}_\varepsilon\right\} \setminus \tilde{E}.$$

Then R' is a noncompact Riemann surface of genus g, and we can choose a marking χ' of R' so that

$$(R', \chi') \subset (\tilde{S}_\varepsilon, \tilde{\sigma}_\varepsilon) \tag{4.5}$$

and that there is a homeomorphism h of (R, χ) onto (R', χ'). We prove that R' and h: $R \to R'$ possess the desired properties. The restriction of $\tilde{\eta}_\varepsilon$ to R is a holomorphic mapping into R' which is homotopic to h. Thus condition (ii) is satisfied. To prove (i) observe that (4.5) implies $[\tilde{S}_\varepsilon, \tilde{\sigma}_\varepsilon] \in M(R', \chi')$. If $[\tilde{S}_\varepsilon, \tilde{\sigma}_\varepsilon]$ belonged to $M(R, \chi)$, then it would follow from (4.1) that

$$\operatorname{Im} \pi_{11}[\tilde{R}, \tilde{\chi}] \leq \operatorname{Im} \pi_{11}[\tilde{S}_\varepsilon, \tilde{\sigma}_\varepsilon]. \tag{4.6}$$

On the other hand, by (4.2) we have $[\tilde{S}_0, \tilde{\sigma}_0] \in M(S_\varepsilon, \sigma_\varepsilon)$. Hence, by (4.3) and (4.4) we obtain

$$\operatorname{Im} \pi_{11}[\tilde{S}_\varepsilon, \tilde{\sigma}_\varepsilon] \leq \operatorname{Im} \pi_{11}[\tilde{S}_0, \tilde{\sigma}_0] = \operatorname{Im} \pi_{11}[\tilde{R}, \tilde{\chi}], \tag{4.7}$$

and conclude from (4.6) and (4.7) that

$$\operatorname{Im} \pi_{11}[\tilde{S}_0, \tilde{\sigma}_0] = \operatorname{Im} \pi_{11}[\tilde{S}_\varepsilon, \tilde{\sigma}_\varepsilon].$$

Thus by virtue of (4.3) we know that $[\tilde{S}_0, \tilde{\sigma}_0]$ is a point of $M(S_\varepsilon, \sigma_\varepsilon)$ such that $\pi_{11}[\tilde{S}_0, \tilde{\sigma}_0]$ has the smallest imaginary part in the disk $\pi_{11}(M(S_\varepsilon, \sigma_\varepsilon))$. But the restriction of $\tilde{\eta}_0$ to S_ε is a conformal mapping of $(S_\varepsilon, \sigma_\varepsilon)$ into $(\tilde{S}_0, \tilde{\sigma}_0)$, the complement of whose image in $\tilde{S}_0$ is of positive area because $S_0 \setminus S_\varepsilon$ is. This violates Proposition 2.4, and the proof of the lemma is complete.

5. Hyperbolic lengths and conformal embeddings

In this section we give a proof of Theorem 1.1. To this end we establish the following theorem, which is interesting in its own right.

THEOREM 5.1: *Let R be a noncompact Riemann surface of positive finite genus with a border-like boundary component. Then there is a noncompact Riemann surface R' together with a homeomorphic bijection $h: R \to R'$ such that*

(i) *there is a holomorphic mapping of R into R' homotopic to h, but*
(ii) *there are no conformal mappings of R into R' homotopic to h.*

Proof: Fix a marking χ of R, and take a marked noncompact Riemann surface (R', χ') and a homeomorphism $h: (R, \chi) \to (R', \chi')$ as in Lemma 3.1. By property (ii) of the lemma there is a holomorphic mapping of R into R' homotopic to h. If there were a conformal mapping of R into R' homotopic to h, then it would be a conformal mapping of (R, χ) into (R', χ') so that $M(R', \chi')$ would be a subset of $M(R, \chi)$, contradicting property (i) of the lemma. ∎

It is now easy to prove Theorem 1.1.

Proof of Theorem 1.1: Let R' and h be as in Theorem 5.1. Since holomorphic mappings decrease hyperbolic lengths, property (i) of Theorem 5.1 implies condition [H]. On the other hand, condition [C] is false by property (ii) of the same theorem. ∎

Remark: Theorem 5.1 is no longer true when the genus is zero. Simply connected Riemann surfaces give simple counterexamples. As a less trivial result we refer to [15, Theorem I], where Schiffer proved that the existence of a holomorphic mapping of a doubly connected Riemann surface into another homotopic to a homeomorphism implies the existence of a conformal embedding homotopic to the homeomorphism. Alternative proofs of the theorem of Schiffer can be found in Huber [4], [5], Jenkins [6], Landau–Osserman [7], [8]. See also Marden–Richards–Rodin [9] and Reich [14]. It is not known whether Theorem 5.1 is true or not for planar Riemann surfaces of connectivity three or more.

In the rest of this section we study the relationship between compact continuations and hyperbolic lengths. Let (R, χ) be a marked noncompact Riemann surface of genus g, where $1 < g < \infty$. For each $c \in \mathcal{S}[R]$ we define a function L_c on the Teichmüller space T_g as follows. For each $[\tilde{R}, \tilde{\chi}] \in T_g$ there is a homeomorphism $\tilde{h}$ of (R, χ) into $(\tilde{R}, \tilde{\chi})$, which induces a mapping $\tilde{h}_*: \mathcal{S}[R] \to \mathcal{S}[\tilde{R}]$. The homotopy class $\tilde{c}[\tilde{R}, \tilde{\chi}] := \tilde{h}_*(c)$ depends only on c, for, any homeomorphism of (R, χ) into $(\tilde{R}, \tilde{\chi})$ induces the same mapping of $\mathcal{S}[R]$ into $\mathcal{S}[\tilde{R}]$. We then define $L_c[\tilde{R}, \tilde{\chi}]$ to be the hyperbolic length of $\tilde{c}[\tilde{R}, \tilde{\chi}]$:

$$L_c[\tilde{R}, \tilde{\chi}] = l_{\tilde{R}}(\tilde{c}[\tilde{R}, \tilde{\chi}]).$$

Now, we denote by $\mathfrak{H}(R,\chi)$ the set of all $[\tilde{R},\tilde{\chi}] \in T_g$ such that

$$L_c[\tilde{R},\tilde{\chi}] \leq l_R(c)$$

for all $c \in \mathcal{S}[R]$.

PROPOSITION 5.1: *The set $\mathfrak{H}(R,\chi)$ is compact.*

Proof: Since L_c is a real analytic function on T_g, we see that $\mathfrak{H}(R,\chi)$ is closed. To show that it is relatively compact, we use the Fenchel-Nielsen coordinate functions on T_g. These coordinate functions are composed of lengths parameters and twist parameters. Observe that $L_c \equiv 0$ or $L_c > 0$ throughout T_g. If $L_c > 0$, then $\tilde{c}[\tilde{R},\tilde{\chi}]$ contains a unique geodesic loop for any $[\tilde{R},\tilde{\chi}] \in T_g$. Moreover, it depends solely on c whether or not this geodesic loop is simple. Now, if L_c is positive and the geodesic loop in $\tilde{c}[\tilde{R},\tilde{\chi}]$ is simple, then there exists $c' \in \mathcal{S}[R]$, depending only on c, with $L_{c'} > 0$ such that the geodesic loop in $\tilde{c}'[\tilde{R},\tilde{\chi}]$ transversally intersects that in $\tilde{c}[\tilde{R},\tilde{\chi}]$. Since $L_{c'}$ is bounded on $\mathfrak{H}(R,\chi)$, it follows from the collar theorem (see, for example, Buser [3, Corollary 4.1.2]) that L_c is bounded away from zero on $\mathfrak{H}(R,\chi)$. Thus each length parameter is bounded above and away from zero on $\mathfrak{H}(R,\chi)$. Furthermore, since the twist parameters are real analytic functions of the lengths of certain closed geodesics (cf. [3, Lemma 3.3.14]), they are also bounded on $\mathfrak{H}(R,\chi)$. Thus $\mathfrak{H}(R,\chi)$ is relatively compact in T_g. ∎

If $[\tilde{R},\tilde{\chi}] \in \mathfrak{M}(R,\chi)$, then there is a holomorphic mapping $\tilde{f}$ of (R,χ) into $(\tilde{R},\tilde{\chi})$ homotopic to a homeomorphic injection $\tilde{h}$: $(R,\chi) \to (\tilde{R},\tilde{\chi})$. Then $\tilde{f}$ induces the same mapping of $\mathcal{S}[R]$ into $\mathcal{S}[\tilde{R}]$ as $\tilde{h}$ does. Since holomorphic mappings decrease hyperbolic lengths, we see that $[\tilde{R},\tilde{\chi}]$ belongs to $\mathfrak{H}(R,\chi)$. We have thus proved that

$$\mathfrak{M}(R,\chi) \subset \mathfrak{H}(R,\chi). \tag{5.1}$$

The next proposition should be compared with Proposition 3.2.

PROPOSITION 5.2: *The set $\mathfrak{M}(R,\chi)$ is a compact subset of T_g if $1 < g < \infty$.*

Proof: By Proposition 5.1 and the inclusion relation (5.1) we have only to verify that $\mathfrak{M}(R,\chi)$ is closed in T_g. We assume from the outset that $(R,\chi) \subset (\tilde{R}_0,\tilde{\chi}_0)$ for some $[\tilde{R}_0,\tilde{\chi}_0] \in T_g$. Let $\{[\tilde{R}_n,\tilde{\chi}_n]\}_{n=1}^{\infty}$ be a sequence in $\mathfrak{M}(R,\chi)$ converging to $[\tilde{R},\tilde{\chi}]$ in T_g. For $n = 0,1,2,\ldots$ there is a Fuchsian group Γ_n acting on the unit disk $\mathbb{D}$ such that $\tilde{R}_n = \mathbb{D}/\Gamma_n$. We also represent $\tilde{R} = \mathbb{D}/\Gamma$ with some Fuchsian group Γ. Every homeomorphism $\tilde{h}_n$: $(\tilde{R}_0,\tilde{\chi}_0) \to (\tilde{R}_n,\tilde{\chi}_n)$ is lifted to a homeomorphism H_n of $\mathbb{D}$ onto itself such that $\pi_n \circ H_n = \tilde{h}_n \circ \pi_0$, where $\pi_n: \mathbb{D} \to \tilde{R}_n$ stands for the natural projection. We may assume that $\{H_n\}$ converges to a homeomorphism $H: \mathbb{D} \to \mathbb{D}$ with $\pi \circ H = \tilde{h} \circ \pi_0$ locally uniformly on $\mathbb{D}$, where $\pi: \mathbb{D} \to \tilde{R}$ is the natural projection and where $\tilde{h}: \tilde{R}_0 \to \tilde{R}$ is a homeomorphic bijection. Let $\theta_n: \Gamma_0 \to \Gamma_n$ (resp. $\theta: \Gamma_0 \to \Gamma$) be the isomorphism defined by $\theta_n(\gamma) = H_n \circ \gamma \circ H_n^{-1}$ (resp. $\theta(\gamma) = H \circ \gamma \circ H^{-1}$) for $\gamma \in \Gamma_0$. Then $\{\theta_n\}$ algebraically converges to θ.

Now, for each n there is a holomorphic mapping $\tilde{f}_n: (R, \chi) \to (\tilde{R}_n, \tilde{\chi}_n)$ homotopic to $\tilde{h}_n|_R$. It is lifted to a holomorphic mapping F_n of $\pi_0^{-1}(R)$ into $\mathbb{D}$ such that $\pi_n \circ F_n = \tilde{f}_n \circ \pi_0$ and $F_n \circ \gamma = \theta_n(\gamma) \circ F_n$ for $\gamma \in \Gamma_0$. We can further require that the points $F_n(0)$, $n = 1, 2, \ldots$, stay in a compact subset of $\mathbb{D}$. Since $\{F_n\}$ is a uniformly bounded sequence of holomorphic functions, it has a subsequence converging to a holomorphic function F locally uniformly on $\pi_0^{-1}(R)$. Clearly, F is a holomorphic mapping of $\pi_0^{-1}(R)$ into $\mathbb{D}$ and satisfies $F \circ \gamma = \theta(\gamma) \circ F$ for $\gamma \in \Gamma_0$. For $z \in \pi_0^{-1}(R)$ and $t \in [0,1]$, define $\varphi(z,t)$ to be the point which divides the hyperbolic line segment joining $F(z)$ to $H(z)$ in the ratio $t : (1-t)$. Then φ is a continuous mapping of $\pi_0^{-1}(R) \times [0,1]$ into $\mathbb{D}$ such that $\varphi(\cdot, 0) = F$ and $\varphi(\cdot, 1) = H$. Since F and H induce the same isomorphism θ of Γ_0 onto Γ, we have $\varphi(\gamma(z), t) = \theta(\gamma)(\varphi(z,t))$ for all $\gamma \in \Gamma_0$, $z \in \pi_0^{-1}(R)$ and $t \in [0,1]$ (see the proof of Ahlfors [1, Lemma on p. 119]). Thus F induces a holomorphic mapping of (R, χ) into $(\tilde{R}, \tilde{\chi})$ homotopic to h, and hence $[\tilde{R}, \tilde{\chi}] \in \mathfrak{M}(R, \chi)$, as desired. ∎

In the above proof, if all f_n are injective, then so are F_n and hence F is conformal by Hurwitz's theorem. This implies that (R, χ) is conformally embedded into $(\tilde{R}, \tilde{\chi})$. We have thus proved the following

PROPOSITION 5.3 (Oikawa [13]): *The set $M(R, \chi)$ is a compact subset of T_g.*

Now, Theorem 3.1 together with (5.1) yields the following result:

THEOREM 5.2: *Let (R, χ) be a marked noncompact Riemann surface of genus g, where $1 < g < \infty$. If R has a border-like boundary component, then $M(R, \chi) \subsetneqq \mathfrak{H}(R, \chi)$.*

Let R be a noncompact Riemann surface of genus g with $1 < g < \infty$, and let $\tilde{R}$ be a compact Riemann surface of the same genus. Fix a homeomorphism $\tilde{h}$ of R into $\tilde{R}$. If

[$\tilde{\mathrm{C}}$] *there exists a conformal mapping of R into $\tilde{R}$ which is homotopic to $\tilde{h}$,*

then

[$\tilde{\mathrm{H}}$] $l_{\tilde{R}}(\tilde{h}_*(c)) \leq l_R(c)$ *for all* $c \in \mathcal{S}[R]$.

Theorem 5.2 shows that condition [$\tilde{\mathrm{H}}$] does not necessarily imply condition [$\tilde{\mathrm{C}}$] (cf. Theorem 1.1).

On the other hand, in the case of genus one, we can characterize condition [$\tilde{\mathrm{C}}$] in terms of extremal lengths of weak homology classes. The **first weak homology group** $H_1^w(W)$ of a Riemann surface W is, by definition, the quotient group of singular 1-cycles on W by the subgroup of dividing cycles. If W is of genus g, then $H_1^w(W)$ is a free abelian group of rank $2g$. Also, if W is compact, then $H_1^w(W)$ coincides with the usual homology group $H_1(W)$. Since every element c of $H_1^w(W)$ is a set of finite unions of closed curves on W, we can speak of its extremal length $\lambda_W(c)$.

Now, the homeomorphism $\tilde{h}$ induces an isomorphism $\tilde{h}_*$ of $H_1^w(R)$ onto $H_1(\tilde{R})$. If condition [$\tilde{\mathrm{C}}$] is valid, then

[$\tilde{\mathrm{E}}$] $\lambda_{\tilde{R}}(\tilde{h}_*(c)) \leq \lambda_R(c)$ *for all* $c \in H_1^w(R)$.

For the converse we have the following theorem (cf. Theorem 1.2):

THEOREM 5.3: *Let R be a noncompact Riemann surface of genus one, and let $\tilde{R}$ be a compact Riemann surface of the same genus. Fix a homeomorphism $\tilde{h}$ of R into $\tilde{R}$. Then condition* [$\tilde{\mathrm{E}}$] *implies condition* [$\tilde{\mathrm{C}}$].

Proof: Take a marking $\chi = \{a, b\}$ of R and set $\tilde{\chi} = \tilde{h}_*(\chi)$. The Teichmüller space T_1 of genus one is identified with the upper half plane $\mathbb{H}$ and then $M(R,\chi)$ is a closed disk (or a point) in $\mathbb{H}$ (cf. Proposition 2.1). Let $\tilde{\tau}$ be the point of $\mathbb{H}$ corresponding to $[\tilde{R}, \tilde{\chi}]$. We are required to prove that condition [$\tilde{\mathrm{E}}$] implies $\tilde{\tau} \in M(R,\chi)$.

Let a^w and b^w be the weak homology classes induced by a and b, respectively. For each $r \in \mathbb{Q}$ let U_r denote the set of $\tau \in \mathbb{H}$ for which

$$\operatorname{Im} \frac{1}{r-\tau} \geq \frac{q^2}{\lambda_R(pa^w - qb^w)},$$

where p and q are coprime integers with $r = p/q$. It then follows from [10, Lemmas 1 and 2] that U_r is a horocycle of $\mathbb{H}$ based at r and inscribes $M(R,\chi)$:

$$\partial U_r \cap \partial M(R,\chi) \neq \varnothing \quad \text{and} \quad U_r \supset M(R,\chi).$$

Note that

$$\operatorname{Im} \frac{1}{r-\tilde{\tau}} = \frac{q^2}{\lambda_{\tilde{R}}(\tilde{h}_*(pa^w - qb^w))}.$$

Therefore, if condition [$\tilde{\mathrm{E}}$] is valid, then we have $\tilde{\tau} \in U_r$ for all $r \in \mathbb{Q}$. Since

$$\bigcup_{r\in\mathbb{Q}} (\partial U_r \cap \partial M(R,\chi))$$

is dense on the boundary of $M(R,\chi)$, this implies that $\tilde{\tau} \in M(R,\chi)$, as required. ∎

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